The mean value of the state vector propagates as

$$\bar{x}_{i+1} = [A-BC]\,[\bar{x}_i + K_i(z_i - H\bar{x}_i)] \tag{12}$$

The mean-square value of the state vector is given by

$$\chi_i = \bar{\chi}_i + M_i \tag{13}$$

where

$$\bar{\chi}_{i+1} = [A-BC]\,[\bar{\chi}_i + M_i - P_i]\,[A-BC]' \tag{14}$$

and

$$M_i = E\{(x_i - \bar{x}_i)(x_i - \bar{x}_i)'\} \qquad P_i = E\{(\hat{x}_i - \bar{x}_i)(\hat{x}_i - \bar{x}_i)'\}$$

$$\chi_i = E\{x_i x_i'\} \qquad \bar{\chi}_i = E\{\bar{x}_i \bar{x}_i'\} \tag{15}$$

For the given system of Eqs. (1) and (5),

$$[A-BC] = [I] \tag{16}$$

which gives

$$\bar{\chi}_{i+1} = \bar{\chi}_i + M_i - P_i \qquad \chi_{i+1} = \bar{\chi}_i + M_i - P_i + M_{i+1} \tag{17}$$

When the system matrices A, B, H and the noise covariances Q, R are constant, the filtering process may reach a steady state. Farrenkoff[2] has shown that this indeed is the case, i.e. when $i \to \infty$,

$$M_i \to M_{i+1} \to M_0 \qquad P_i \to P_0 \qquad K_i \to K_0 \tag{18}$$

Hence, from Eqs. (17)

$$\chi_{i+1} - \chi_i = \bar{\chi}_{i+1} - \bar{\chi}_i = M_0 - P_0 \tag{19}$$

and since the prefilter covariance M_0 is larger than the postfilter covariance P_0, the mean-square values of the state variables at update intervals constitute sequences monotonically increasing in time. This gives rise to an instability of the combined estimator-regulator system, with the result that the attitude determination accuracy will degrade over time.

Conclusion

The known procedure of using the drift rate estimate of a rate-integrating gyroscope for drift correction leads to an unstable infinite-time regulator estimation problem. This fact must be considered when finalizing the sensor specifications to be used in the configuration discussed, so that system accuracy remains within the specified bounds during the expected life of the mission.

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Output Ensemble Average of Periodically Excited Linear Time-Varying Systems

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Introduction

IN this Note, we consider a linear time-varying system with a periodical random phase input. One of the properties to describe the output of such system is its mean square ensemble average. The computation of this value is required, inter alia, in the analysis of periodical random phase maneuvers of an airplane evading from a guided missile.[1] In a recent paper,[2] it was shown for long missile flight times (exceeding 8-10 guidance time constants) that the miss distance can be considered as a stationary random variable, and the ensemble average can be computed as a mean square time average.[3-5] For short flight times, where the output is clearly not stationary, only the ensemble average has a meaning. In the past, the analysis of linear time-varying systems driven by a periodical random process was mainly based on Monte Carlo methods.[6]

In this paper, a direct method for computing the mean square ensemble average of the output of such a system is presented. The result can be obtained either analytically or numerically in a single computer run. The considerable reduction in computational effort makes the method very attractive for intensive system analysis. The applicability of the direct method is demonstrated in an example of a missile intercept problem.

Problem Formulation

Consider a linear, single input-single output, casual time-varying system characterized by its impulse response function $g(t,\theta)$. The system input $X(t)$ and output $Y(t)$ are related by

$$Y(t) = \int_0^t g(t,\theta) X(\theta)\, d\theta \tag{1}$$

The type of input to be considered is a periodical one

$$X(t) = A \sin(\omega t + \phi_i) \tag{2}$$

and the initial phase ϕ_i is a random variable with probability density function.

$$p_{\phi_i}(\beta) = \begin{cases} 1/2\pi & (-\pi \le \beta \le \pi) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

In the general case, $Y(t)$ will be a random nonstationary process. The statistical property of interest considered in this paper is the mean square ensemble average of $Y(t)$.

Mean Square Ensemble Average Calculation

Lemma 1: Let a linear time-varying system defined by Eq. (1) be driven by a periodical input with random phase. The mean square ensemble average of the (nonstationary) output

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of the system is given by

$$E\{Y^2(t)\} = [G_1^2(t) + G_2^2(t)]/2 \tag{4}$$

where

$$G_1(t) = A\int_0^t g(t,\theta)\sin(\omega\theta)\,d\theta \tag{5}$$

$$G_2(t) = A\int_0^t g(t,\theta)\cos(\omega\theta)\,d\theta \tag{6}$$

Proof: Substitution of Eq. (2) into Eq. (1) results in

$$Y(t) = A\int_0^t g(t,\theta)\sin(\omega\theta + \phi_i)\,d\theta$$

$$= A\int_0^t g(t,\theta)\,[\sin(\omega\theta)\cos(\phi_i) + \cos(\omega\theta)\sin(\phi_i)]\,d\theta \tag{7}$$

Since ϕ_i is a random constant, Eq. (7) can be rewritten as

$$Y(t) = A\Big\{\cos\phi_i \int_0^t g(t,\theta)\,\sin(\omega\theta)\,d\theta$$

$$+ \sin\phi_i \int_0^t g(t,\theta)\cos(\omega\theta)\,d\theta\Big\} \tag{8}$$

Substitution of Eq. (5) into Eq. (8) leads to

$$Y(t) = \cos\phi_i G_1(t) + \sin\phi_i G_2(t) \tag{9}$$

from which the mean square ensemble average can be calculated[3] by

$$E\{Y^2(t)\} = \int_{-\infty}^{\infty} \alpha^2 p_Y(\alpha)\,d\alpha \tag{10}$$

where $p_Y(\alpha)$ is the probability density function of $Y(t)$. Since $Y(t)$ is a function of ϕ_i, the fundamental theorem of expectation[3] can be used:

$$E\{Y^2(t)\} = \int_{-\infty}^{\infty} Y^2(t,\beta)\cdot p_{\phi_i}(\beta)\,d\beta \tag{11}$$

Substitution of Eqs. (3) and (9) into Eq. (11) results in

$$E\{Y^2(t)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} [\cos\beta\cdot G_1(t) + \sin\beta\cdot G_2(t)]^2\,d\beta \tag{12}$$

The operations of squaring and integration on β lead to Eq. (4), completing the proof.

Remarks:

1) The method is not limited to probability density functions given by Eq. (3).

2) Equation (4) is our main result; $G_1(t)$ and $G_2(t)$ are the system responses to sine and cosine inputs, respectively.

3) $G_1(t)$ and $G_2(t)$ can be derived either in a closed form or by numerical integration. Even if the second approach has to be taken, due to complexity of $g(t,\theta)$, only a single computer run is needed to generate the mean square ensemble average as a function of time. This is the essential contribution of the direct method presented in this paper.

4) This method can be equally used in adjoint analysis.[5,7] For this application, appropriate "shaping filters,"[2] with sine and cosine impulse responses, have to be added.

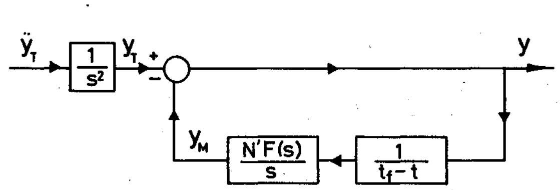

Fig. 1 Two-dimensional proportional navigation—simplified block diagram.

Extension to Generalized Periodic Inputs

Consider a periodic input with random initial phase and with probability density functions given by Eq. (3). Such an input can be represented by its Fourier series expansion.

$$X(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega t + \phi_n) \tag{13}$$

Following the same procedure as in the previous paragraph, it can be shown that in this case the mean square ensemble average is given by

$$E\{Y^2(t)\} = \tfrac{1}{2}\sum_{n=1}^{\infty} b_n^2 [G_{1n}^2(t) + G_{2n}^2(t)] \tag{14}$$

where

$$G_{1n} = \int_0^t g(t,\theta)\sin(n\omega\theta)\,d\theta \tag{15}$$

$$G_{2n} = \int_0^t g(t,\theta)\cos(n\omega\theta)\,d\theta \tag{16}$$

Example

We demonstrate the usefulness of the direct method by considering the interception of a maneuvering target by a homing missile.[1,8] The simplified block diagram of a missile guided by proportional navigation homing against a maneuvering target is shown in Fig. 1. The target performs a periodical evasive maneuver described by a sinusoidal variation of its lateral acceleration perpendicular to the line-of-sight.

$$\ddot{y}_T(t) = a_T \sin(\omega t + \phi_i) \tag{17}$$

The initial phase is a random variable with a probability function defined by Eq. (3). For such a random evasive maneuver, the average miss distance of interest is the root mean square ensemble average of the relative missile target displacement $y(t)$ at the predicted final time t_f:

$$(\bar{m}^2)^{1/2} \triangleq [E\{y^2(t_f)\}]^{1/2} \tag{18}$$

Defining $h(t,\theta)$ as the miss distance impulse response of the guidance loop for target maneuver, the average miss distance is given by

$$(\bar{m}^2)^{1/2} = (1/\sqrt{2})\,[H_1^2(t_f) + H_2^2(t_f)]^{1/2} \tag{19}$$

where

$$H_1(t_f) = a_T\int_0^{t_f} h(t,\theta)\sin(\omega\theta)\,d\theta \tag{20}$$

$$H_2(t_f) = a_T\int_0^{t_f} h(t,\theta)\cos(\omega\theta)\,d\theta \tag{21}$$

In general, $h(t,\theta)$ is not known in a closed form; the system equations have to be integrated numerically.

In order to save computer time and to generate the miss distances for all values of t_f in a single computer run, adjoint analysis[5,7] is applied. The normalized average miss distance

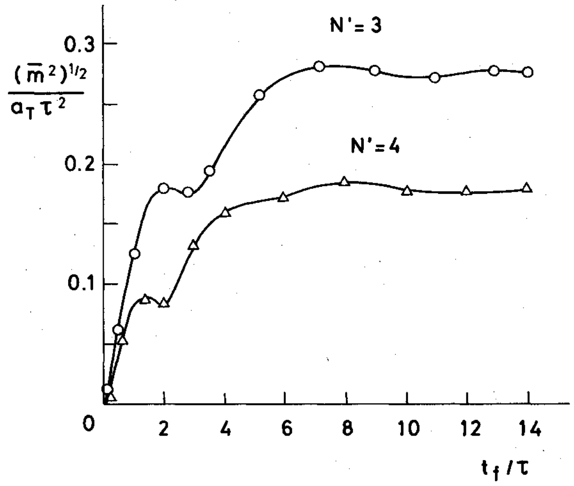

Fig. 2 Normalized average (rms) miss distance.

for a first-order guidance transfer function

$$F(s) = 1/(1+\tau s) \tag{22}$$

is shown as a function of normalized time of flight in Fig. 2. The results were compared to the average of 50 Monte Carlo runs using the actual maneuver process, showing less than 1% of difference for short time of flight. Results for large values of t_f were identical to previously published data.

Conclusions

The direct method presented in this paper for the calculation of the mean square ensemble average of nonstationary functions can be of great use in the analysis of linear time-varying systems with periodical random phase inputs. The results can be obtained either by a closed-form solution or by a single computer run instead of numerous Monte Carlo simulations. The method can be equally applied for adjoint analysis.

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Recursive Parameter Identification for Nonlinear Stochastic Processes

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Introduction

SEVERAL methods[1-7] have been presented for obtaining least-squares estimates of unknown parameters of systems modeled by nonlinear differential equations with discrete measurements made on their response. The unidentified parameters are estimated by minimizing a mean square performance index. Convergence of the iterative algorithms of quasilinearization of Bellman,[2] the Newtonian iteration procedure of Goodman,[1] the parametric differentiation of Chapman and Kirk,[3] and the continuation method of Wasserstrom[4] are all dependent on a good initial guess of the parametric vector. Although convergence and the initial estimate of the parametric vector are closely related for any iterative procedure, choice of the computational algorithm is of paramount importance in affecting the rate of convergence. In the present analysis the nonlinear process and observation equations are linearized and cast into standard linear forms in terms of state, parametric, and observation difference vectors. The Kalman filter algorithm is then employed to obtain recursive estimates of the state difference and parametric difference vectors. The recursive estimate algorithm of the state and parametric vectors is then derived.

Analysis

Consider a nonlinear discrete-time stochastic process represented by

$$x_k = \phi(x_{k-1}, \xi, u_{k-1}) + w_k \qquad w_k \sim N(0, Q_k) \tag{1a}$$

$$z_k = h(x_k) + v_k \qquad v_k \sim N(0, R_k) \tag{1b}$$

where x is an n-dimensional state vector, u is an m-dimensional control vector, z is a p-dimensional observation vector, and ξ is the true r-dimensional parametric vector which is not known a priori. w and v are the process and measurement noise vectors, respectively.

Denoting the estimate of the parametric vector at step $(k-1)$ by ξ_{k-1}, and its estimate error by γ_{k-1}, Eq. (1a) may be written in the form

$$x_k = \phi(x_{k-1}, \xi_{k-1}, u_{k-1}) + \Pi_{k-1}\gamma_{k-1} + w_k \tag{2}$$

where

$$\Pi_{k-1} = (\partial\phi/\partial\xi)_{k-1} \qquad \gamma_{k-1} = \xi - \xi_{k-1} \tag{3}$$

Using the differencing approach,[8] Eq. (2) may be cast into the convenient linear form

$$s_k = \Phi_{k-2}s_{k-1} + (\Pi_{k-1} - \Pi_{k-2})\gamma_{k-1} + U_{k-2}c_{k-1} + \nu_k \tag{4}$$

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